Analysis and Simulation of a Continuum Fluid Dynamics Model for the Vocal Folds

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Abstract

We derive reduced one-dimensional model equations on the vocal fold motion from the two dimensional compressible Navier-Stokes equations coupled with an elastic damped driven wave equation on the fold cover (epithelium), and analyze the oscillation modes of the linearized system about a flat fold. We found two kinds of oscillation modes; one exists when the initial velocity and pressure exceed a threshold proportional to the fold damping, and the other when the initial pressure and velocity exceed a constant times fluid viscosity. We also analyzed the reduced system under the quasi-steady approximation and compared the vocal fold equation in the small vibration regime with that of the Titze model. Our model shares the main qualitative features of the oscillation onset conditions with the Titze model yet differs in the specific form of energy feedback from the air flow to the fold motion. Our model allows arbitrary fold shape and treats the fold as a continuum; in contrast, the Titze model assumes a linear fold shape among other ad hoc working assumptions and concerns only the midpoint motion of the fold.

1 Mathematical Model

We model the vocal fold as a finite mass elastic tube of cross sectional area $A(x,t)$. For example, we can take rectangular cross section, length $2L$, width $2w$, and variable height $2h$, then $A = 4wh$. Alternatively, we may take an elliptical shape with principal axes of length $2h = 2h(x,t)$, and $2L$. The air flows from $x = -L$ to $x = L$. We introduce our one dimensional continuum model, then give a derivation from the two dimensional isentropic Navier-Stokes equations. The vocal flows are low Mach number and free of shocks, and so the isentropic assumption is valid. The model consists of reduced aerodynamic equations and elastic wave equations. The aerodynamic equations are:

• conservation of mass:

$$(A\rho)_t + (\rho\, u\, A)_x = 0, \tag{1.1}$$

ρ air density, u air velocity;

• conservation of momentum:

$$u_t + uu_x = -\frac{1}{\rho}p_x + \frac{A_t\, u}{A} + \frac{A_t\, u}{A} + \frac{4\mu}{3\rho}A^{-1}(A\, u_x)_x, \tag{1.2}$$

p air pressure, μ air viscosity.

The force balance on the fold (tube wall) gives:

• the dynamic boundary motion:

$$m(A - A_{eq})_{tt} = \sigma(A - A_{eq})_{xx} - \alpha(A - A_{eq})_t - \beta(A - A_{eq}) + Sp + f_m. \tag{1.3}$$

Here: m is the fold mass density; σ is the longitudinal elastic tension of the fold; α is the muscle damping constant; β is an elastic modulus modeling the vibration property of the fold in the vertical; S is a cross section shape related factor, $S = 4w$ for rectangular cross sections, and $S = \pi w$ for elliptical cross sections; f_m is a prescibed function to model muscle tone so that a particular fold shape A_{eq} (converging or diverging or flat) serves as an equilibrium state.

We shall write (1.3) into:

$$mA_{tt} = \sigma A_{xx} - \alpha A_t - \beta A + Sp + \tilde{f}_m, \tag{1.4}$$

where $\tilde{f}_m = \tilde{f}_m(x)$ is prescribed forcing. For our analysis, we shall normalize $m = 1$ and ignore the tilde.

• The equation of state:

$$p = \kappa \, \rho^{\gamma}, \; \gamma > 1, \; \kappa > 0, \tag{1.5}$$

the system (1.1)-(1.5) is closed, and we solve an initial boundary value problem on $x \in [-L, L]$ with proper in flow boundary conditions and initial fold shape.

In studying collapsible tubes ([12], [8]) and air flow through duct of spatially varying cross section [15], it is common to use (1.1) with the one dimensional unsteady Euler equation which is (1.2) with the last two terms omitted. The major differences in the two modeling problems are: (1) a vocal fold is fast oscillatory in time (e.g. 100 - 200 Hz), (2) the vocal fold carries mass, and a *dynamic (damped driven wave) equation* is necessary to describe the fold motion; moreover, the vocal fold has mechanical damping. In contrast, collapsible tubes are massless and damping free [12]. The conservation of mass and momentum in the collapsible tube system is recovered when $A_t \ll A$ and $\mu \to 0$. In [12], the tube cross section is related to the pressure p by a tube law (analogue of equation of state: $p = A^{n_1} - A^{n_2}$) with $\rho = 1$. Here, ρ and p are related by the equation of state, the gamma gas law; then the cross section A is related to p *dynamically.*

We derive the fluid part of the model system assuming that the fold varies in space and time as $A = A(x, t)$. Consider a two dimensional slightly viscous subsonic air flow in a channel with spatially temporally varying cross section in two space dimensions, $\Omega_0 = \Omega_0(t) = \{(x, y) : x \in [-L, L], y \in [-A(x, t)/2, A(x, t)/2]\}$, where $A(x, t)$ denotes the channel width with a slight abuse of notation, or cross sectional area if the third dimension is uniform and equal to one. The two dimensional Navier-Stokes equations in differential form are [2]:

• conservation of mass:

$$\rho_t + \nabla \cdot (\rho \, \vec{u}) = 0; \tag{1.6}$$

• conservation of momentum:

$$(\rho \vec{u})_t = -\nabla \cdot (\rho \, (\vec{u} \otimes \vec{u}) + \int_{\partial \Omega} \sigma \cdot \vec{n} \, dS; \tag{1.7}$$

where σ is the stress tensor, $\sigma = (\sigma_{ij}) = -p\delta_{ij} + d_{ij}$, and:

$$d_{ij} = 2\mu \, (e_{ij} - \frac{div \vec{u}}{3} \delta_{ij}), \quad e_{ij} = \frac{1}{2}(u_{i,x_j} + u_{j,x_i}), \quad (x_1, x_2) \equiv (x, y);$$

μ is the fluid viscosity; Ω is any volume element of the form:

$$\Omega = \{(x, y) : x \in [a, b] \subset [-L, L], y \in [-A(x,t)/2, A(x,t)/2].\}.$$

The equation of state is (1.5).

The boundary conditions on $(\rho, \vec{u})$ are:

(1) on the upper and lower boundaries $y = \pm A(x,t)/2$, $\rho_y = 0$, and $\vec{u} = (0, \pm A_t/2)$, the velocity no slip boundary condition;

(2) at the inlet, $\vec{u}(-L, y, t) = \vec{u}_l$, a prescribed inlet velocity, $\rho(-L, y, t) = \rho_l$, a prescribed inlet density (deduced from input pressure);

(3) at the outlet, $\vec{u}_x(L, y, t) = 0$, $\rho_x(L, y, t) = \rho_{atm} - \rho(L, y, t)$, to minimize reflected waves.

We are only concerned with flows that are symmetric in the vertical. For positive but small viscosity, the flows are laminar in the interior of Ω_0 and form viscous boundary layers near the upper and lower edges. The vertically averaged flow quantities are expected to be much less influenced by the boundary layer behavior.

Let us assume that the flow variables obey:

$$\begin{aligned} |u_{1,y}| \ll |u_{1,x}|, \; |u_{2,y}| \ll |u_{1,x}|, & \text{ away from boundaries of } \Omega_0, \\ |\vec{u}_{\perp}| \gg |\vec{u}_{||}|, & \text{ near the boundaries of } \Omega_0, \\ |\rho_y| \ll |\rho_x|, & \text{ throughout } \Omega_0. \end{aligned} \tag{1.8}$$

These are consistent with empirical observations in the viscous boundary layers [2], namely, there are large vertical velocity gradients, yet small pressure or density gradients in the boundary layers. The boundary layers are of width $O(\mu^{1/2})$. Denote by $\overline{\rho}$, $\overline{u}_1$, the vertical averages of ρ and u_1. Note that the exterior normal $\vec{n} = (-A_x/2, 1)/(1 + A_x^2/4)^{1/2}$ if $y = A/2$, $\vec{n} = (-A_x/2, -1)/(1 + A_x^2/4)^{1/2}$ if $y = -A/2$.
Let $a = x$, $b = x + \delta x$, $\delta x \ll 1$, and start with the identity:

$$\frac{d}{dt}\int_{\Omega(t)} \rho\, dV = \frac{d}{dt}\int_{\Omega(t_0)} \rho\, J(t)\, dV = \int_{\Omega(t_0)} \rho_t\, J(t)\, dV + \int_{\Omega(t_0)} \rho\, J_t\, dV, \quad (1.9)$$

where $J(t)$ is the Jacobian of volume change from a reference time t_0 to t. Since $\Omega(t)$ is now a thin slice, $J(t) = \frac{A(t)}{A(t_0)}$ for small δx, and $J_t = A_t(t)/A(t_0)$. The second integral in (1.9) is:

$$\int_{\Omega(t_0)} \rho\, J_t\, dV = \rho \frac{A_t(t)}{A(t_0)} A(t_0)\delta x = \overline{\rho}\, A_t(t)\delta x. \quad (1.10)$$

The first integral is simplified using (1.6) as:

$$\int_{\Omega(t_0)} \rho_t\, J(t)\, dV = \int_{\Omega(t)} \rho_t\, dV = -\int_{\partial\Omega(t)} \rho\, \vec{u}\cdot\vec{n}\, dS. \quad (1.11)$$

We calculate the last integral of (1.11) further as follows:

$$\begin{aligned}
\int_\Omega \rho\vec{u}\cdot\vec{n}\, ds &= \int_{-A/2}^{A/2} (-\rho u_1)(x, y, t)\, dy + \int_{-A/2}^{A/2} (\rho u_1)(x + \delta x, y, t)\, dy \\
&+ \int_x^{x+\delta x} \rho\cdot(0, A_t/2)\cdot(-A_x/2, 1)\, dx \\
&+ \int_x^{x+\delta x} \rho\cdot(0, -A_t/2)\cdot(-A_x/2, -1)\, dx \\
&= \overline{\rho u_1} A|_x^{x+\delta x} + \frac{\delta x}{2}(\rho\, A_t)_{y=A/2} + \frac{\delta x}{2}(\rho\, A_t)_{y=-A/2} + O((\delta x)^2) \\
&\approx (\overline{\rho}\cdot\overline{u}_1 A)|_x^{x+\delta x} + \overline{\rho} A_t \delta x + O((\delta x)^2), \quad (1.12)
\end{aligned}$$

where we have used the smallness of ρ_y to approximate $\rho|_{y=\pm A/2}$ by $\overline{\rho}$ and $\overline{\rho u_1}$ by $\overline{\rho}\cdot\overline{u_1}$.

Combining (1.9)-(1.11), (1.12) with:

$$\frac{d}{dt}\int_{\Omega} \rho \, dV = (\overline{\rho} A \delta x)_t + O((\delta x)^2), \tag{1.13}$$

dividing by δx and sending it to zero, we have:

$$(\overline{\rho} A)_t + (\overline{\rho} \cdot \overline{u}_1 A)_x = 0,$$

which is (1.1).

Next consider $i = 1$ in the momentum equation, $a = x$, $b = x + \delta x$. We have similarly with (1.7):

$$\begin{aligned} & \frac{d}{dt}\int_{\Omega(t)} \rho \, u_1 \, dV = \int_{\Omega(t)} (\rho u_1)_t \, dV + \int_{\Omega(t_0)} \rho \, u_1 \, J_t \, dV \\ = & \; - \int_{\partial\Omega(t)} \rho u_1 \vec{u} \cdot \vec{n} \, dS + \int_{\partial\Omega(t)} \sigma_{1,j} \cdot \vec{n}_j \, dS \\ + & \; \int_{\Omega(t)} - \, dV + \overline{\rho} \, \overline{u_1} \, A_t \delta x. \end{aligned} \tag{1.14}$$

We calculate the integrals of (1.14) below.

$$\frac{d}{dt}\int_{\Omega} \rho u_1 \, dV = (\overline{\rho u}_1 A)_t \delta x + O((\delta x)^2) \approx (\overline{\rho} \cdot \overline{u} A)_t \cdot \delta x + O((\delta x)^2),$$

$$\int_{\partial\Omega} \rho u_1 \vec{u} \cdot \vec{n} \, dS = (\overline{\rho} \cdot \overline{u}_1^2 A)|_x^{x+\delta x} + O(\mu^{1/2}), \tag{1.15}$$

where the smallness of $u_{1,y}$ in the interior and small width of boundary layer $O(\mu^{1/2})$ gives the $O(\mu^{1/2})$ for approximating $\overline{u_1^2}$ by $\overline{u_1} \cdot \overline{u_1}$.

$$\begin{aligned} \int_{\partial\Omega} - p \delta_{1,j} n_j dS & \approx -\overline{p} A|_x^{x+\delta x} + \int_x^{x+\delta x} p \, A_x \, dx \\ & = -\overline{p} A|_x^{x+\delta x} + \overline{p} \, A_x \, \delta x + O((\delta x)^2). \end{aligned}$$

By assumptions (1.8), $div \, \vec{u} \approx u_{1,x}$ away from boundary layers, hence:

$$d_{1,j} \approx 2\mu(e_{1j} - e_{11}\delta_{1,j}/3),$$

away from boundary layers. Near boundary layers, $\nabla \vec{u} = O(\mu^{-1/2})$, and $d_{1,1} = O(\mu^{1/2})$. It follows that $\overline{d_{11}} \approx \frac{4}{3}\mu\overline{e_{11}} + O(\mu^{3/2})$. Thus the contribution from the left and right boundaries located at x and $x+\delta x$ is:

$$
\begin{aligned}
\sum_{l,r} \int_{l,r} d_{11} n_1 &= A\,\overline{d_{11}}|_x^{x+\delta x} \approx \frac{4}{3} A\,\mu\,\overline{e_{11}}|_x^{x+\delta x} + O(\mu^{3/2})\,\delta x \\
&\approx \frac{4\mu}{3}(A\overline{u_1}_x)|_x^{x+\delta x} + O(\mu^{3/2})\,\delta x. \qquad (1.16)
\end{aligned}
$$

The contribution from the upper and lower boundaries is:

$$
\begin{aligned}
\sum_{\pm} \int_{y=\pm A/2} d_{11}\, n_1\, dS &= -d_{11} A_x \delta x/2|_{y=A/2} - d_{11} A_x \delta x/2|_{y=-A/2} \\
&= \mu\delta x \sum_{\pm} O(\partial_\perp \vec{u})|_{y=\pm A/2}. \qquad (1.17)
\end{aligned}
$$

Similarly,

$$
\sum_{\pm} \int_{y=\pm A/2} d_{12} n_2\, dS = \mu\delta x \sum_{\pm} O(\partial_\perp \vec{u})|_{y=\pm A/2}. \qquad (1.18)
$$

Since $\partial_\perp \vec{u}|_{y=\pm A/2} = O(\mu^{-1/2})$, the viscous flux from the boundary layers are $O(\mu^{1/2})$, much larger than the averaged viscous term $\frac{4\mu}{3}(A\overline{u_1}_x)_x = O(\mu)$ after sending $\delta x \to 0$, though they all converge to zero as $\mu \to 0$, away from the wake turbulent region. We notice that the vertically averaged quantities have little dependence on the viscous boundary layers. Hence the quantities from upper and lower edges in (1.17) and (1.18), and that in (1.15), should balance themselves. Omitting them altogether, and combining remaining terms that involve only $\overline{u_1}$, $\overline{p}$ in the bulk, we end up with:

$$
(\overline{\rho} \cdot \overline{u_1} A)_t + (\overline{\rho} \cdot \overline{u_1}^2 A)_x = -(\overline{p}\,A)_x + A_x \overline{p} + \frac{4\mu}{3}(A\overline{u_1}_x)_x. \qquad (1.19)
$$

Simplifying (1.19) with the continuity equation (1.1), we find that:

$$
\overline{u_1}_t + \overline{u_1}\,\overline{u_1}_x = -\overline{p}_x/\overline{\rho} + \frac{A_t \overline{u_1}}{A} + \frac{4\mu}{3\overline{\rho}} A^{-1}(A\overline{u_1}_x)_x + \overline{\rho u_1}\,A_t, \qquad (1.20)
$$

which is (1.2).

2 Linear Stability Analysis near a Flat Fold

In this section, we discuss the existence of a neutral oscillation mode of the linearized system around a flat fold. It turns out that such a mode exist for system (1.1)-(1.5) because of the $A_t\, u/A$ term or the viscous term. The presence of oscillation mode due to the balance of dissipation in fluids with the loss term of elastic media is known in subsonic flutter [5] for $\nu > 0$. In our case here, it is essential that the derivation originated with the no slip boundary condition on the fold, which implies that enough energy of the background flow can transfer to the fold to offset the loss there. We show this phenomenon below with a stability analysis.

The system (1.1)-(1.5) admits constant steady states: (u_0, p_0, A_0), $p_0 = \kappa\rho_0^\gamma$, satisfying:

$$-\beta A_0 + 4w^2 p_0 = f_m, \tag{2.1}$$

where f_m is a constant so that A_0 matches the height of the connecting vocal tract. The constant states here may not be very realistic in that there is a normally a curved transition from vocal tract to vocal fold, however we ignore this effect for the moment. The constant states are pertinent to Y. Qi's experimental demonstration (Nov, 98) where an elastic membrane (with thickness and mass density) is mounted on solid tubes, and air is pumped through the membrane channel with enough velocity and pressure to induce vibration and sound.

We are interested in conditions leading to the small amplitude oscillations near the constant steady states. This is similar to Titze [13], where a lumped ODE is proposed and analyzed for the fold center using mucosal wave approximation. However here, we perform calculations directly from (1.1)-(1.5), and do not make any further modeling assumptions. Our analysis will be extended to the regime where a background pressure gradient is present. Since we have zero background pressure gradient, our results do not compare directly yet with [13], though some qualitative features such as the threshold conditions are similar.

Letting $u = u_0 + \hat{u}$, $p = p_0 + \hat{p}$, $\rho = \rho_0 + \hat{\rho}$, $A = A_0 + \hat{A}$, and linearizing, we get ($\nu = \frac{4\mu}{3\rho_0}$):

$$(A_0\hat{\rho} + \hat{A}\rho_0)_t + (\rho_0 A_0 \hat{u} + \rho_0 u_0 \hat{A} + u_0 A_0 \hat{\rho})_x = 0, \tag{2.2}$$

$$\hat{u}_t + u_0 \hat{u}_x + \frac{1}{\rho_0}\hat{p}_x = \nu \hat{u}_{xx} + \frac{u_0}{A_0}\hat{A}_t, \tag{2.3}$$

$$\hat{A}_{tt} = \sigma \hat{A}_{xx} - \alpha \hat{A}_t - \beta \hat{A} + 4w^2 \hat{p}. \tag{2.4}$$

Equations (2.2)-(2.3) are written as:

$$A_0\hat{\rho}_t + \rho_0 \hat{A}_t + \rho_0 A_0 \hat{u}_x + \rho_0 u_0 \hat{A}_x + u_0 A_0 \hat{\rho}_x = 0, \tag{2.5}$$

$$\hat{u}_t + u_0 \hat{u}_x + \frac{\hat{p}_x}{\rho_0} = \nu \hat{u}_{xx} + \frac{u_0}{A_0}\hat{A}_t. \tag{2.6}$$

Applying the operator $\partial_t + u_0 \partial_x - \nu \partial_{xx}$ on (2.5) and using (2.6), we find:

$$(\partial_t + u_0\partial_x - \nu\partial_{xx})(A_0\hat{\rho}_t + \rho_0\hat{A}_t + \rho_0 u_0 \hat{A}_x + u_0 A_0 \hat{\rho}_x) - A_0 \hat{p}_{xx} + u_0 \rho_0 \hat{A}_t = 0. \tag{2.7}$$

Differentiating (1.5) gives: $p_t = \kappa\gamma\rho^{\gamma-1}\rho_t$, or $\hat{p}_t = \kappa\gamma\rho_0^{\gamma-1}\hat{\rho}_t$ upon linearizing at $\rho = \rho_0$. Similarly, $\hat{p}_x = \kappa\gamma\rho_0^{\gamma-1}\hat{\rho}_x$. With these relations, equation (2.7) becomes:

$$\begin{aligned} &\Gamma(\partial_t + u_0\partial_x - \nu\partial_{xx})(\partial_t + u_0\partial_x)\hat{p} - \hat{p}_{xx} + \frac{u_0\rho_0}{A_0}\hat{A}_t \\ + \quad & A_0^{-1}\rho_0(\partial_t + u_0\partial_x - \nu\partial_{xx})(\partial_t + u_0\partial_x)\hat{A} = 0, \end{aligned} \tag{2.8}$$

where:

$$\Gamma = \frac{1}{\kappa\gamma\rho_0^{\gamma-1}} = \frac{1}{c^2},$$

with c being the speed of sound at air density ρ_0.

Applying the operator $A_0\Gamma(\partial_t + u_0\partial_x - \nu\partial_{xx})(\partial_t + u_0\partial_x) - A_0\partial_{xx}$ to both sides of (2.4), we get:

$$A_0[\Gamma\partial_{tt} + 2\Gamma u_0\partial_{xt} + (\Gamma u_0^2 - 1)\partial_{xx} - \nu\Gamma\partial_{xxt} - \Gamma\nu u_0\partial_{xxx}] \cdot (\hat{A}_{tt} - \sigma\hat{A}_{xx} + \alpha\hat{A}_t + \beta\hat{A})$$

$$= -4w^2\rho_0[(\partial_t + u_0\partial_x)(\partial_t + u_0\partial_x) - \nu\partial_{xxt} - \nu u_0\partial_{xxx}\hat{A} + u_0\rho_0\hat{A}_t]. \qquad (2.9)$$

Substituting the mode $\hat{A} = A_m e^{im'x+\lambda t}$, $m' = \frac{m\pi}{L}$, we end up with the following algebraic equation of degree four for λ:

$$[\Gamma\lambda^2 + 2\Gamma u_0 m'\lambda i + (1-\Gamma u_0^2)m'^2 + \nu\Gamma m'^2\lambda + \Gamma\nu u_0 i m'^3]\cdot[\lambda^2 + \sigma m'^2 + \alpha\lambda + \beta]$$

$$= -4w^2\rho_0 A_0^{-1}[\lambda^2 + 2u_0 i m'\lambda - u_0^2 m'^2 + \nu m'^2\lambda + i\nu u_0 m'^3] - \frac{4w^2 u_0\rho_0\lambda}{A_0}, \quad (2.10)$$

or:

$$\Gamma\lambda^4 + (\alpha\Gamma + 2\Gamma u_0 m' i + \nu\Gamma m'^3)\lambda^3 +$$

$$(\Gamma(\beta+\sigma m'^2) + \alpha(2\Gamma u_0 m' i + \nu\Gamma m'^2) + 4w^2\rho_0 A_0^{-1} + (1-\Gamma u_0^2)m'^2 + \Gamma\nu u_0 m'^3 i)\lambda^2$$

$$+((2\Gamma u_0 m' i + \nu\Gamma m'^2)(\beta + \sigma m'^2) - \alpha m'^2(\Gamma u_0^2 - 1)$$

$$+\alpha\Gamma\nu u_0 i m'^3 + 4w^2\rho_0 A_0^{-1}(2u_0 i m' + \nu m'^2) + \frac{4w^2 u_0\rho_0}{A_0})\lambda$$

$$+[(\beta + \sigma m'^2)((1-\Gamma u_0^2)m'^2 + \Gamma\nu u_0 m'^3 i) + 4w^2\rho_0 A_0^{-1}(\nu u_0 i m'^3 - u_0^2 m'^2)] = 0. \qquad (2.11)$$

Proposition 2.1 *Let $\nu = 0$ in (2.11). (1) If*

$$\rho_0\, u_0^2 > \alpha(\rho_0\, u_0 + \Gamma\beta\frac{u_0 A_0}{4w^2}), \qquad (2.12)$$

(2.11) has a pure imaginary solution, implying the existence of an oscillation mode.

(2) If the Mach number $M \equiv u_0/c \in (0,1)$ is small, $\alpha = O(u_0)$, $\Gamma\beta \gg M$, $\Gamma\sigma \gg M$, then there is an oscillatory mode of the form:

$$|m'| = \beta\left(\frac{2w^2\rho_0}{A_0} + \Gamma\beta\right)\left(-\beta + \sigma(2w^2\rho_0 A_0^{-1} + \Gamma\beta)\right)^{-1} + O(|M|),$$

if $2w^2\rho_0\sigma > A_0\beta$ and

$$\frac{2u_0\rho_0}{A_0\alpha} = \frac{2w^2\rho_0}{A_0} + \Gamma\beta.$$

Proof: (1) Let $\nu = 0$, and $\lambda = i\eta$ in (2.11), where η is real. The real and imaginary parts give:

$$\begin{aligned}\Gamma\eta^4 &+& 2\Gamma u_0 m'\eta^3 - [\Gamma(\sigma m'^2 + \beta) + m'^2(1 - \Gamma u_0^2) + \frac{4w^2\rho_0}{A_0}]\eta^2 \\ &+& [-2\Gamma u_0 m'(\beta + \sigma m'^2) - \frac{8w^2\rho_0 u_0 m'}{A_0}]\eta \\ &+& [m'^2(1 - \Gamma u_0^2)(\sigma m'^2 + \beta) - \frac{4u_0^2 m'^2 w^2 \rho_0}{A_0}] = 0, \end{aligned} \tag{2.13}$$

and:

$$-\alpha\Gamma\eta^3 - 2\alpha\Gamma u_0 m'\eta^2 + m'^2\alpha(1 - \Gamma u_0^2)\eta + \frac{4w^2 u_0 \rho_0 \eta}{A_0} = 0. \tag{2.14}$$

For $\eta \neq 0$, $\alpha \neq 0$, we have from (2.14):

$$\Gamma\eta^2 + 2\Gamma u_0 m'\eta - m'^2(1 - \Gamma u_0^2) - \frac{4w^2 u_0 \rho_0}{A_0 \alpha} = 0,$$

so:

$$\eta = -u_0 m' \pm c\sqrt{m'^2 + 4w^2 u_0 \rho_0/(A_0\alpha)}. \tag{2.15}$$

Now we regard the left hand side of (2.13) as a continuous function of m', call it $F(m')$. For $|m'| \gg 1$, $\eta \sim (-u_0 \pm c)m'$, direct calculation shows:

$$F(m') \sim -\frac{4w^2\rho_0}{\Gamma A_0} m'^2 < 0.$$

While for $|m'| \ll 1$,

$$\eta \sim \pm 2w\sqrt{\frac{u_0\rho_0}{\Gamma A_0 \alpha}} + O(m'),$$

and:

$$F(m') \sim \frac{4w^2 u_0 \rho_0}{\Gamma A_0 \alpha}\left(\frac{4w^2 u_0 \rho_0}{A_0 \alpha} - \Gamma\beta - \frac{4w^2\rho_0}{A_0}\right) > 0,$$

provided:

$$\frac{4w^2 u_0 \rho_0}{A_0} > \alpha\left(\frac{4w^2\rho_0}{A_0} + \Gamma\beta\right), \tag{2.16}$$

holds. Under (2.16), $F(m)=0$ has a nonzero real solution, hence a pure imaginary solution exists to (2.11) when $\nu=0$.

(2) Let us turn to the small Mach number and damping regime where more explicit oscillatory modes can be obtained. Consider $\epsilon = u_0/c \ll 1$, $\Gamma\beta = O(1)$, $\Gamma\sigma = O(1)$, $u_0 = O(\alpha)$. Then $\eta \sim c\eta_1 = \pm c\sqrt{m'^2 + \frac{4w^2 u_0 \rho_0}{A_0 \alpha}}$, $\eta_1 = O(1)$. Plugging this scaling into (2.11) with $\nu = 0$, dropping $O(\epsilon)$ and higher order terms, we find:

$$\eta_1^4 - (\Gamma(\sigma m'^2 + \beta) + m'^2 + \frac{4w^2\rho_0}{A_0})\eta_1^2 + m'^2\Gamma(\sigma m'^2 + \beta) = 0, \tag{2.17}$$

which simplifies into:

$$(\frac{u_0\rho_0\Gamma\sigma}{A_0\alpha} + \frac{\rho_0}{A_0} - \frac{u_0\rho_0}{A_0\alpha})m'^2 = \frac{u_0\rho_0}{A_0\alpha}(\frac{4w^2u_0\rho_0}{A_0\alpha} - \frac{4w^2\rho_0}{A_0} - \Gamma\beta). \tag{2.18}$$

Let us select with order one ratio $\frac{u_0}{\alpha}$:

$$\frac{2w^2u_0\rho_0}{A_0\alpha} = \frac{2w^2\rho_0}{A_0} + \Gamma\beta, \tag{2.19}$$

also satisfying (2.16). Then (2.18) defines a nonzero $|m'|$:

$$|m'|^2 = \beta\left(\frac{2w^2\rho_0}{A_0} + \Gamma\beta\right)\left(-\beta + \sigma(\frac{2w^2\rho_0}{A_0} + \Gamma\beta)\right)^{-1}. \tag{2.20}$$

if in addition:

$$2w^2\rho_0\sigma > A_0\beta. \tag{2.21}$$

If $2w^2\rho_0\sigma$ is close to $A_0\beta$, the $|m'|$ in (2.20) can be nearly $\frac{\beta}{\sigma}(1+\frac{1}{\sigma\Gamma})$. We end the proof.

Remark 2.1 *Condition (2.16) says that the initial velocity and pressure must be large enough to overcome the damping constant α. A similar calculation on system (1.1)-(1.5) shows that $F(m') = -4w^2\rho_0 m'^2/(\Gamma A_0) < 0$ for all m', implying non-existence of oscillatory modes. We see the viscous effect coming from the fold boundary of the original two dimensional problem.*

Condition (2.16) is similar to the threshold pressure in Titze's model [13] in the sense that minimum energy (analogous to minimum lung pressure) must exceed the fold damping coefficient times the prephonatory half width A_0 times another constant (compare with the second term on the right hand side of (2.12)).

Remark 2.2 *There is another neutral yet nonoscillatory mode from (2.13)-(2.14), namely, $\eta = 0$, and:*

$$\sigma |m_0'|^2 = \frac{4u_0^2 w^2 \rho_0}{(1-M^2)A_0} - \beta, \tag{2.22}$$

provided the right hand side expression is positive.

Next we turn on $\mu > 0$ in (1.2), and consider (2.11) with $0 < \nu \ll 1$. The oscillation mode in Proposition 2.1 will generically be slightly perturbed and preserves its oscillation nature. The second neutral nonoscillatory mode however will be perturbed into a new oscillation mode as we show below.

Proposition 2.2 *If $\nu > 0$, and the right hand side of (2.22) is positive, there is an additional oscillation mode of the form:*

$$\begin{aligned} \eta \;\; = \;\; & \nu\eta_1 = -\nu[\alpha m'^2(1-\Gamma u_0^2) + \frac{4w^2 u_o \rho_0}{A_0}]^{-1} \\ & \cdot [(\beta + \sigma m'^2)\Gamma u_0 m'^3 + 4w^2 \rho_0 A_0^{-1} u_0 m'^3] + O(\nu^2), \end{aligned} \tag{2.23}$$

where:

$$\begin{aligned} m' = m_0' + \nu \left(4u_0^2 w^2 \rho_0 \, A_0^{-1} - (1-\Gamma u_0^2)(\beta + 2\sigma (m')_0^2)\right)^{-1} \\ \cdot \left(\Gamma u_0(\beta + \sigma m_0^2) + 4w^2\rho_0 A_0^{-1} u_0\right) \cdot \left(\alpha (m_0')^2 (1-\Gamma u_0^2) + \frac{4w^2 u_0 \rho_0}{A_0}\right)^{-1} \\ \cdot \left((\beta + \sigma (m_0')^2)\Gamma u_0 (m_0')^3 + 4w^2 \rho_0 A_0^{-1} u_0 (m_0')^3\right) + O(\nu^2). \end{aligned} \tag{2.24}$$

Proof: Write down the real and imaginary parts of equation (2.11) with $\lambda = i\eta$:

$$\Gamma\eta^4 + 2\Gamma u_0 m'\eta^3 - (\Gamma(\beta + \sigma m'^2) + \alpha\nu\Gamma m'^2 + 4w^2\rho_0 A_0^{-1} + (1 - \Gamma u_0^2)m'^2)\eta^2$$
$$+(-2\Gamma u_0 m'(\beta + \sigma m'^2) - \alpha\Gamma\nu u_0 m'^3 - 8w^2\rho_0 A_0^{-1} u_0 m')\eta$$
$$+(\beta + \sigma m'^2)(1 - \Gamma u_0^2)m'^2 - 4w^2\rho_0 A_0^{-1} u_0^2 m'^2 = 0, \qquad (2.25)$$

and:

$$-(\alpha\Gamma + \nu\Gamma m'^3)\eta^3 - (2\alpha\Gamma u_0 m' + \Gamma\nu u_0 m'^3)\eta^2$$
$$+(\nu\Gamma m'^2(\beta + \sigma m'^2) - \alpha m'^2(\Gamma u_0^2 - 1) + 4w^2\rho_0 A_0^{-1}\nu m'^2 + \frac{4w^2 u_0\rho_0}{A_0})\eta$$
$$+(\beta + \sigma m'^2)(\Gamma\nu u_0 m'^3) + 4w^2\rho_0 A_0^{-1}\nu u_0 m'^3 = 0. \qquad (2.26)$$

Looking for $\eta = \nu\eta_1 + O(\nu^2)$ in (2.26), and keeping $O(\nu)$ terms, we find (2.23). Plugging (2.23) into (2.25) and seeking $m' = m'_0 + \nu m'_1 + O(\nu^2)$, we find (2.24) after some algebra, and end the proof.

Remark 2.3 *This second oscillatory mode is similar to the subsonic flutter anomaly found in [5], where the von Kármán plate equation models the flexible walls and the oscillation frequency appears due to a positive small* ν*. However, the mode here exists even in the limit* $\alpha \to 0$*.*

The condition that the right hand side of (2.22) be positive requires flow velocity u_0 *or pressure* p_0 *or density* ρ_0 *to be large enough to exceed* β *for given* A_0 *and* w*. This is very different from the requirment of (2.16) which involves* α*.*

3 Quasi-steady Approximation and Relations with the Titze Model

The glottal flow is nearly quasisteady and inviscid if the fold opening is not too small, and the resulting Bernoulli's law is often adopted as an approximation, [6], [13] among others. In this approximation, the temporal variation of flow variables is considered much slower than that of the fold motion. Thus let us drop the time derivatives, and viscous term in the flow equations to get:

$$(\rho\, u\, A)_x = 0, \tag{3.1}$$

$$u\, u_x = -\frac{p_x}{\rho} + \frac{A_t\, u}{A}, \tag{3.2}$$

while keeping the fold dynamic equation and the equation of state the same.

Our goal is to derive a closed equation for the cross section area A. Integrating (3.1) and (3.2) in x using the equation of state shows:

$$\rho\, u\, A = Q_0, \tag{3.3}$$

$$u^2/2 + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} = \int_{-L}^{x} \frac{A_t\, u}{A}\, dx\, + P_0, \tag{3.4}$$

where P_0 and Q_0 are constants determined by the flow conditions at the inlet $x = -L$. Note that without the integral term with $\frac{A_t\, u}{A}$, (3.4) becomes the standard Bernoulli's law.

Substituting (3.3) into (3.4) gives:

$$\frac{Q_0^2}{2\rho^2\, A^2} + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} = Q_0 \int_{-L}^{x} \frac{A_t}{\rho\, A^2}\, dx\, + P_0. \tag{3.5}$$

We would obtain a closed equation on A if (3.5) could be solved for ρ in terms of A. If A_t is small (small fold vibration), we see that (3.5) has no positive solution ρ for a sufficiently small A. This shows that the quasisteady approximation is not valid for the general situation of small fold openings, in which case the viscous term must be taken into account.

On the other hand, if A undergoes small vibration about a constant state A_0 which satisfies:

$$\frac{Q_0^2}{2\rho_0^2\, A_0^2} + \frac{\gamma \rho_0^{\gamma-1}}{\gamma - 1} = P_0,$$

then (3.5) can be solved for ρ by perturbation. Letting $A = A_0 + \hat{A}$, $\rho = \rho_0 + \hat{\rho}$, we find:

$$-\frac{Q_0^2}{\rho_0^3\, A_0^2}\hat{\rho} - \frac{Q_0^2}{\rho_0^2 A_0^3}\hat{A} + \gamma \rho_0^{\gamma-2}\hat{\rho} - Q_0 \int_{-L}^{x} \frac{\hat{A}_t}{A_0^2 \rho_0} = 0,$$

or:

$$\hat{\rho}\cdot\left(\gamma\rho_0^{\gamma-2}-\frac{Q_0^2}{\rho_0^3\,A_0^2}\right)=\frac{Q_0^3}{A_0^2\rho_0}\hat{A}+\frac{Q_0}{A_0^2\rho_0}\int_{-L}^{x}\hat{A}_t. \tag{3.6}$$

If $\gamma\rho_0^{\gamma-2}>\frac{Q_0^2}{\rho_0^3\,A_0^2}$, or:

$$\rho_0^{\gamma+1}>\frac{Q_0^2}{\gamma\,A_0^2}, \tag{3.7}$$

which means large pressure or small enough cross section area for given Q_0, then:

$$\hat{\rho}=\frac{Q_0^2}{\rho_0 A_0^2}\cdot\frac{1}{\gamma\rho_0^{\gamma-2}-\frac{Q_0^2}{\rho_0^3\,A_0^2}}\left[\int_{-L}^{x}\hat{A}_t+\frac{Q_0}{\rho_0 A_0}\hat{A}\right]. \tag{3.8}$$

Now substituting (3.8) into the fold dynamic A equation, we obtain:

$$\hat{A}_{tt}-\sigma\hat{A}_{xx}+\alpha\hat{A}_t+\beta\hat{A}=c_{aero}[\int_{-L}^{x}\hat{A}_t+\frac{Q_0}{\rho_0 A_0}\hat{A}], \tag{3.9}$$

where:

$$c_{aero}=\frac{4w\kappa\gamma\rho_0^{\gamma-1}}{\gamma\rho_0^{\gamma-2}-\frac{Q_0^2}{\rho_0^3\,A_0^2}}\frac{Q_0^2}{\rho_0 A_0^2}>0. \tag{3.10}$$

Equation (3.9) can be written as:

$$\hat{A}_{tt}-\sigma\hat{A}_{xx}+(\alpha\hat{A}-c_{aero}\int_{-L}^{x}\hat{A})_t+(\beta-c_{aero}Q_0\rho_0^{-1}A_0^{-1})\hat{A}=0. \tag{3.11}$$

Equation (3.11) is a linear wave equation with damping and pumping. This is easy to see from the energy identity. Assuming that $\hat{A}$ is periodic in x, we have:

$$\begin{aligned}&\frac{d}{dt}\int_{-L}^{L}dx\,\left(\frac{1}{2}\hat{A}_x^2+\frac{\sigma}{2}\hat{A}_t^2+(\beta-c_{aero}Q_0\rho_0^{-1}A_0^{-1})\hat{A}^2/2\right)=\\&-\alpha\int_{-L}^{L}dx\,\hat{A}_t^2+c_{aero}(\int_{-L}^{L}dx\,\hat{A}_t)^2/2.\end{aligned} \tag{3.12}$$

We shall require that:

$$\beta-c_{aero}Q_0\rho_0^{-1}A_0^{-1}>0, \tag{3.13}$$

which is true if ρ_0 is sufficiently large for fixed β and Q_0. Then (3.12) says that the rate of change of the total energy in time depends on the balance of the natural fold damping (the negative term with prefactor α) and the energy input from the aerodynamic flow (the positive term with prefactor c_{aero}).

To find an oscillatory mode in (3.11), define $\varphi = \int_{-L}^{x} \hat{A}$, so φ satisfies:

$$\varphi_{xtt} - \sigma\varphi_{xxx} + \alpha\varphi_{xt} - c_{aero}\varphi_t + (\beta - c_{aero}Q_0\rho_0^{-1}A_0^{-1})\varphi_x = 0.$$

We insert Fourier mode of the form $\varphi = \varphi_k e^{i\pi kx/L}$, and get:

$$\begin{aligned}(ik\pi/L)\varphi_{k,tt} &+ (\alpha(ik\pi/L) - c_{aero})\varphi_{k,t} \\ &+ (ik\pi/L)(\sigma(ik\pi/L)^2 + \beta - c_{aero}Q_0/(\rho_0\, A_0))\varphi_k = 0,\end{aligned}$$

or:

$$\varphi_{k,tt} + (\alpha + iLc_{aero}/(k\pi))\varphi_{k,t} + (-\sigma\, k^2\pi^2/L^2 + \beta - c_{aero}Q_0/(\rho_0\, A_0))\varphi_k = 0. \tag{3.14}$$

The ODE (3.14) has an oscillatory solution if:

$$Lc_{aero}(k\pi)^{-1} \gg \alpha = O(1). \tag{3.15}$$

Noticing from (3.10) that for large ρ_0, $c_{aero} \sim O(A_0^{-2})$, hence (3.15) can be satisfied if A_0^2 is small enough for given α and L. The overall condition for creating an oscillation mode is that for given Q_0, L, β, and α, choose p_0 (or ρ_0) sufficiently large and A_0 sufficiently small so that

$$\rho_0^{\gamma+1} \gg A_0^{-2}, \quad 4wL\kappa\gamma Q_0^2 A_0^{-2} \gg \alpha. \tag{3.16}$$

These two restrictions imply:

$$\kappa\gamma\rho_0^{\gamma+1} \gg \alpha Q_0^{-2}/(4wL). \tag{3.17}$$

If we regard the left hand side as an analogue of pressure, then (3.17) says that the pressure has to be above a certain threshold which is proprotional to damping constant α and inversely proportional to L, the fold longitudinal length. These requirements agree qualitatively with those of the Titze model [13] we briefly review and compare with next.

Titze [13] assumed that during oscillation, the vocal fold cover propagates a surface mucosal wave in the direction of the air flow and the body of the vocal ligament and muscle is steady. The fold shape is approximated as a straight line connecting the fold entry of height h_1 and the fold exit of height h_2. Taylor expanding a mucosal wave with constant velocity c, Titze [13] approximated entry area A_1 and exit area A_2 as:

$$A_1 = 2w(h_0 + \hat{h} + \tau \hat{h}_t), \; A_2 = 2w(h_0 + \hat{h} - \tau \hat{h}_t), \tag{3.18}$$

where $\tau = 2L/c$ the travel time for the mucosal wave to reach exit from entrance. The fold motion is lumped onto the midpoint $x = 0$, and postulated as ($h = h_0 + \hat{h}$):

$$m h_{tt} + \alpha h_t + \beta h = \frac{1}{2L} \int_{-L}^{L} P(x)\, dx \equiv P_g. \tag{3.19}$$

Using Bernoulli's law and linear fold shape, [13] showed that in particular for small fold vibration about rectangular fold equilibrium position A_0:

$$P_g = P_s \left(1 - \frac{A_2}{A_1}\right) = \frac{2\tau P_s \hat{h}_t}{h_0 + \hat{h} + \tau \hat{h}_t}, \tag{3.20}$$

where P_s is the subglottal pressure, and exit pressure is zero. Linearizing (3.20) about h_0, and keeping only linear terms, one has:

$$m\hat{h}_{tt} + (\alpha - 2\tau P_s h_0^{-1})\hat{h}_t + \beta \hat{h} = 0, \tag{3.21}$$

from which follows the threshold pressure condition:

$$\alpha - 2\tau P_{s,*} h_0^{-1} = 0, \; P_{s,*} = h_0 c \alpha / (4L). \tag{3.22}$$

So the oscillation threshold pressure is proprotional to damping constant α, inversely proportional to fold longitudinal length $2L$; moreover, small opening makes the formation of oscillation easier.

Both our model and the Titze model captures the positive energy feedback from air flow into the fold motion, though they are not of the same form. Our model admits a systematic treatment free of ad hoc assumptions in the Titze model; moreover, the fold shape is more realistic and allows arbitrary spatial variation.

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